Bayesian decision making

Václav Hlaváč

Czech Technical University in Prague
Faculty of Electrical Engineering, Department of Cybernetics
Center for Machine Perception
http://cmp.felk.cvut.cz/~hlavac, hlavac@fel.cvut.cz

Courtesy: M.I. Schlesinger

Outline of the talk:

- Bayesian task formulation.
- Two general properties of the Bayesian task.
- Probability of the wrong estimate.
- Reject option.
- Non-random strategy is good.
- Linear separability in space of probabilities, convex cones.

Notation remarks
Joint and conditional probabilities

- The joint probability $p_{XY}(x, y)$ can be expressed as $p_{XY}(x, y) = p_{X_y}(x|y) \cdot p_Y(y)$.
- The standard notation for joint and conditional probabilities is ambiguous.
  - Are $p(x, y)$ and $p(x|y)$ numbers, functions of a single variable or functions of two variables?
  - Let us disambiguate using subscripts:
    $p_{XY}(x, y)$ is a *function of two variables*,
    $p_{X_y}(x|y)$ is a *function of a single variable* x,
    and $p_{xy}(x, y)$ is a *single real number*.

Bayesian decision making, concepts

Object (situation) is described by two parameters:

- x is an observable feature (observation).
- y is an unobservable hidden parameter (state).

- X is a finite set of observations, $x \in X$.
- Y is a finite set of hidden states, $y \in Y$.
- D is a finite set of possible decisions $d \in D$.

- $p_{XY}\colon\ X \times Y \to \mathbb{R}$ is the joint probability that the object is in the state y and the observation x is made.
- $W\colon\ Y \times D \to \mathbb{R}$ is a *penalty function*, $W(y,d)$, $y \in Y$, $d \in D$ is the penalty paid in for the object in the state y and the decision d made.
- $q\colon\ X \to D$ is a *decision function* (rule, strategy) assigning to each $x \in X$ the decision $q(x) \in D$.
- $R(q)$ is the risk, i.e. the mathematical expectation of the penalty.

Definition of the Bayesian decision making task

- Given: sets X, Y and D, a joint probability $p_{XY}\colon X \times Y \to \mathbb{R}$ and function $W\colon Y \times D \to \mathbb{R}$
- Task: The Bayesian task of statistical decision making seeks a strategy $q\colon X \to D$ which minimizes the Bayesian risk

$$R(q) = \sum_{x \in X} \sum_{y \in Y} p_{XY}(x, y)\, W(y, q(x))\,.$$

The solution to the Bayesian task is the Bayesian strategy q minimizing the risk.

The formulation can be extended to infinite X, Y and D by replacing summation with integration and probability with probability density.

Example: two hidden states, three decisions

- Object: a patient examined by a physician.
- Observations (some measurable parameters): $X = \{$temperature, blood pressure, $\dots\}$.
- Two unobservable states: $Y = \{$healthy, sick$\}$.
- Three decisions: $D = \{$not cured, weak medicine, strong medicine$\}$.
- The penalty function: $W : Y \times D \to \mathbb{R}$.

$W(y,d)$	not cured	weak medicine	strong medicine
sick	10	2	0
healthy	0	5	10

Comments on the Bayesian decision making (1)

In the Bayesian decision making (recognition):

- Decisions do not influence the state of nature (unlike, e.g. in game theory, control theory).
- A single decision is made, issues of time are ignored in the model (unlike in control theory, where decisions are typically taken continuously and are expected in a real-time).
- The cost of obtaining measurements is not modelled (unlike in the sequential decision theory).

Comments on the Bayesian decision making (2)

The hidden parameter y (the class information) is considered not observable.

Common situations are:

- y could be observed but only at a high cost.
- y is a future state (e.g., the predicted petrol price) and will be observed later.

It is interesting to ponder whether a state can ever be genuinely unobservable (cf. Schrödinger's cat).

Classification is a special case of the decision-making problem where the set of decisions D and hidden states Y coincide.

Generality of the Bayesian formulation

Observation x can be a number, symbol, function of two variables (e.g., an image), graph, algebraic structure, etc.

Application	Measurement	Decisions
value of a coin in a slot machine	$x \epsilon \mathbb{R}^n$	value
optical character recognition	2D bitmap, gray-level image	words, numbers
license plate recognition	gray-level image	characters, numbers
fingerprint recognition	2D bitmap, gray-level image	personal identity
speech recognition	$x(t)$	words
EEG, ECG analysis	$\bar{x}(t)$	diagnosis
forfeit detection	various	{yes, no}
speaker identification	$x(t)$	personal identity
speaker verification	$x(t)$	{yes, no}

Two general properties of Bayesian strategies

1. Deterministic strategies are always better than randomized ones.
2. Each Bayesian strategy corresponds to separation of the space of probabilities into convex subsets.

Before proving the above mentioned general properties, let us explain two practically useful special cases of Bayesian decision making tasks.

Two particular Bayesian tasks

1. Probability of the wrong estimate of the state.

   In most cases, the pattern recognition task is to estimate the state of an object. This means that a set of decisions D and a set of states Y are the same.

2. Decision with the reject option, i.e., `not known`.

   The task is to estimate the state of an object with a high confidence or to reject the decision.

Probability of the wrong estimate of the state (1)

The decision $q(x) = y$ means that an object is in the state y. The estimate $q(x)$ not always is equal to the actual state y^*. Thus the probability of the wrong decision $q(x) \neq y^*$ is required to be as small as possible.

A unit penalty is paid the situation $q(x) \neq y^*$ occurs and no penalty is paid otherwise,

$$W\big(y^*, q(x)\big) = \begin{cases} 0 & \text{if} \quad q(x) = y^* \, , \\ 1 & \text{if} \quad q(x) \neq y^* \, . \end{cases}$$

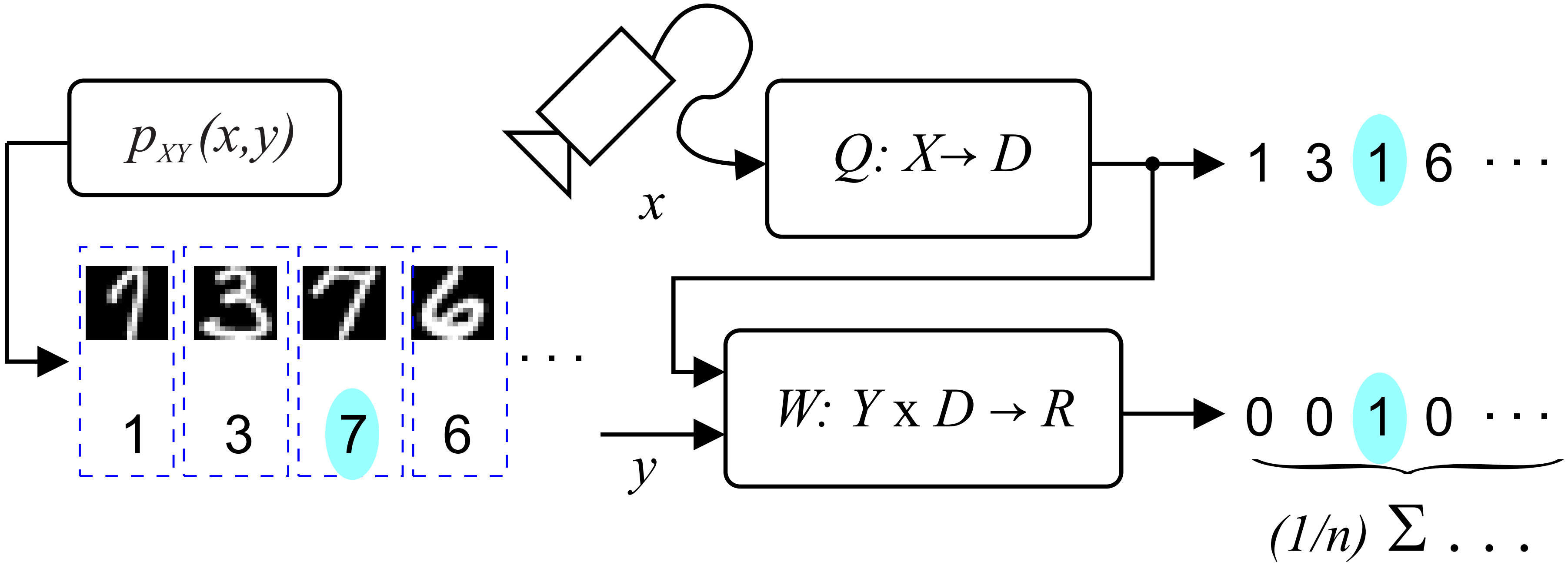
The Bayesian risk

$$R(q) = \sum_{x \in X} \sum_{y^* \in Y} p_{XY}(x, y^*) \; W\big(y^*, q(x)\big)$$

becomes the probability of the wrong estimate of the state $q(x) \neq y^*$.

Example: optical character recognition

Illustration of the Bayesian setting:



- X – set of all possible intensity images.
- Y – set of numerals $\{0, 1, \ldots, 9\}$.
- D – equals to Y, i.e., decision assigns images to classes.
- W – 0/1-loss function $W(y, q(x)) = \begin{cases} 0 & \text{if} \quad y = q(x) \, , \\ 1 & \text{if} \quad y \neq q(x) \, . \end{cases}$

Probability of the wrong estimate of the state (2)

We have to determine the strategy $q\colon X \to Y$ which minimizes the risk, i.e.,

$$\begin{aligned}
q(x) &= \operatorname*{argmin}_{y\in Y} \sum_{y^*\in Y} p_{XY}(x\,,y^*)\; W(y^*,y) \\
&= \operatorname*{argmin}_{y\in Y} \sum_{y^*\in y} p_{Y|X}(y^*\,|\,x)\; p(x)\; W(y^*,y) \\
&= \operatorname*{argmin}_{y\in Y} \sum_{y^*\in y} p_{Y|X}(y^*\,|\,x)\; W(y^*,y) \\
&= \operatorname*{argmin}_{y\in Y} \sum_{y^*\in Y\setminus\{y\}} p_{Y|X}(y^*\,|\,y) \\
&= \operatorname*{argmin}_{y\in Y} \left(\sum_{y^*\in Y} p_{Y|X}(y^*\,|\,x) - p_{Y|X}(y\,|\,x) \right) \\
&= \operatorname*{argmin}_{y\in Y} \big(1 - p_{Y|X}(y\,|\,x)\big) = \operatorname*{argmax}_{y\in Y}\, p_{Y|X}(y\,|\,x)\,.
\end{aligned}$$

The result is that the *a posteriori* probability of each state y is to be calculated for the observation x and it is to be decided in favor of the most probable state.

Bayesian strategy with a reject option (1)

Let us introduce a conditional mathematical expectation of the penalty by $R(x,d)$, called a partial risk (also a conditional risk) given the observation x,

$$R(x,d) = \sum_{y \in Y} p_{Y|X}(y \mid x)\, W(y,d)\,.$$

- Bayesian risk equals $R(q) = \sum_{x \in X} p_X(x)\, R(x, q(x))$.
- Decision $d = q(x)$ has to correspond to the minimal partial risk $R(x,d)$.
- Sometimes this minimum will be quite large and the resulting decision should be `not known`.
- Decision `not known` is given if the observation x does not contain enough information to decide with a small risk.

Bayesian strategy with a reject option (2)

Let X and Y be sets of observations and states, $p_{XY}\colon X \times Y \to \mathbb{R}$ be a joint probability distribution and $D = Y \cup \{\texttt{not known}\}$ be a set of decisions.

Let us set penalties $W(y,d)$, $y \in Y$, $d \in D$:

$$W(y,d) = \begin{cases} 0, & \text{if } d = y\,, \\ 1, & \text{if } d \neq y \text{ and } d \neq \texttt{not known}\,, \\ \varepsilon, & \text{if } d = \texttt{not known}\,. \end{cases}$$

Task: Find the Bayesian strategy $q\colon X \to D$ such that the decision $q(x)$ corresponding to the observation x has to minimize the partial risk,

$$q(x) = \operatorname*{argmin}_{d \in D} \sum_{y^* \in Y} p_{Y|X}(y^* \mid x)\, W(y^*, d)\,.$$

Bayesian strategy with a reject option (3)

The equivalent definition of the Bayesian strategy

$$q(x) = \begin{cases} \operatorname*{argmin}_{d \in Y} R(x,d)\,, & \text{if } \min_{d \in Y} R(x,d) < R(x, \texttt{not known})\,, \\ \texttt{not known}\,, & \text{if } \min_{d \in Y} R(x,d) \geq R(x, \texttt{not known})\,. \end{cases}$$

There holds for $\min_{d \in Y} R(x,d)$

$$\begin{aligned} \min_{d \in Y} R(x,d) &= \min_{d \in Y} \sum_{y^* \in Y} p_{Y|X}(y^* \mid x)\, W(y^*, d) \\ &= \min_{y \in Y} \sum_{y^* \in Y \setminus \{y\}} p_{Y|X}(y^* \mid x) \\ &= \min_{y \in Y} \left(\sum_{y^* \in Y} p_{Y|X}(y^* \mid x) - p_{Y|X}(y \mid x) \right) \\ &= \min_{y \in Y} \big(1 - p_{Y|X}(y \mid x)\big) = 1 - \max_{y \in Y} p_{Y|X}(y \mid x)\,. \end{aligned}$$

Bayesian strategy with a reject option (4)

There holds for $R(x, \texttt{not known})$

$$R(x, \texttt{not known}) = \sum_{y^* \in Y} p_{Y|X}(y^* \,|\, x)\, W(y^*, \texttt{not known})$$
$$= \sum_{y^* \in Y} p_{Y|X}(y^* \,|\, x)\, \varepsilon = \varepsilon\,.$$

The decision rule becomes

$$q(x) = \begin{cases} \operatorname*{argmax}_{y \in Y} p_{Y|X}(y \,|\, x)\,, & \text{if } 1 - \max_{y \in Y} p_{Y|X}(y \,|\, x) < \varepsilon\,, \\ \texttt{not known}\,, & \text{if } 1 - \max_{y \in Y} p_{Y|X}(y \,|\, x) \geq \varepsilon\,. \end{cases}$$

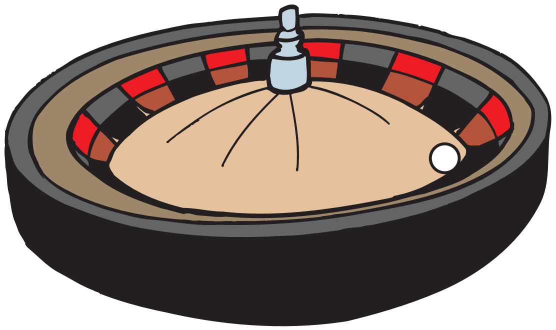
Bayesian strategy with a reject option (5)

Bayesian strategy with the reject option $q(x)$ in words:

- The state y has to be found which has the largest *a posteriori* probability.
- If this probability is larger than $1 - \varepsilon$ then it is decided in favor of the state y.
- If this probability is not larger than $1 - \varepsilon$ then the decision `not known` is provided.

Q: Does it make sense to randomize Bayesian strategies?

$x \longrightarrow$ $q\colon X \times D \rightarrow R$ $\longrightarrow$



$\xrightarrow{q_r}$

Answer: **No!**

A deterministic strategy is never worse than a randomized one.

Bayesian strategies are deterministic

Instead of $q\colon X \to D$, consider a stochastic strategy (probability distributions) $q_r(d \,|\, x)$.

Theorem

Let X, Y, D be finite sets, $p_{XY}\colon X \times Y \to \mathbb{R}$ be a probability distribution, $W : Y \times D \to \mathbb{R}$ be a penalty function. Let $q_r\colon D \times X \to \mathbb{R}$ be a stochastic strategy. Its risk is

$$R_{\text{rand}} = \sum_{x \in X} \sum_{y \in Y} p_{XY}(x, y) \sum_{d \in D} q_r(d \,|\, x)\, W(y, d)\,.$$

In such a case, there exist a deterministic (Bayesian) strategy $q\colon X \to D$ with the risk

$$R_{\text{det}} = \sum_{x \in X} \sum_{y \in Y} p_{XY}(x, y)\, W\big(y, q(x)\big)$$

which is not greater than R_{rand}.

Proof (Bayesian strategy is deterministic)

$$R_{\mathrm{rand}} = \sum_{x\in X} \sum_{d\in D} q_r(d\,|\,x) \sum_{y\in Y} p_{XY}(x,y)\, W(y,d)\,.$$

$\sum_{d\in D} q_r(d\,|\,x) = 1,\; x \in X, \quad q_r(d\,|\,x) \geq 0,\; d \in D,\; x \in X\,.$

$$R_{\mathrm{rand}} \geq \sum_{x\in X} \min_{d\in D} \sum_{y\in Y} p_{XY}(x,y)\, W(y,d) \quad \text{holds for all } x \in X,\; d \in D\,. \tag{1}$$

Let us denote by $q(x)$ any value d that satisfies the equality

$$\sum_{y\in Y} p_{XY}(x,y)\, W\big(y, q(x)\big) = \min_{d\in D} \sum_{y\in Y} p_{XY}(x,y)\, W(y,d)\,. \tag{2}$$

The function $q\colon X \to D$ defined in such a way is a deterministic strategy which is not worse than the stochastic strategy q_r. In fact, when we substitute Equation (2) into the inequality (1) then we obtain the inequality

$$R_{\mathrm{rand}} \geq \sum_{x\in X} \sum_{y\in Y} p_{XY}(x,y)\, W\big(y, q(x)\big)\,.$$

The right hand side gives the risk of the deterministic strategy q. $R_{\mathrm{det}} \leq R_{\mathrm{rand}}$ holds.

A special case: two states only, likelihood ratio

- Hidden state assumes two values only, $Y = \{1, 2\}$.
- Only conditional probabilities $p_{X|1}(x)$ and $p_{X|2}(x)$ are known.
- The *a priori* probabilities $p_Y(1)$ and $p_Y(2)$ and penalties $W(y, d)$, $y \in \{1, 2\}$, $d \in D$, are not known.
- In this situation, the Bayesian strategy cannot be created.
- Nevertheless, the strategy cannot be an arbitrary one and should follow certain constraints.

Likelihood ratio (2)

If the *a priori* probabilities $p_Y(y)$ and the penalty $W(y,d)$ were known then the decision $q(x)$ about the observation x ought to be

$$\begin{aligned} q(x) &= \operatorname*{argmin}_d \big(p_{XY}(x,1)\, W(1,d) + p_{XY}(x,2)\, W(2,d)\big) \\ &= \operatorname*{argmin}_d \big(p_{X|1}(x)\, p_Y(1)\, W(1,d) + p_{X|2}(x)\, p_Y(2)\, W(2,d)\big) \\ &= \operatorname*{argmin}_d \left(\frac{p_{X|1}(x)}{p_{X|2}(x)}\, p_Y(1)\, W(1,d) + p_Y(2)\, W(2,d)\right) \\ &= \operatorname*{argmin}_d \big(\gamma(x)\, c_1(d) + c_2(d)\big)\ . \end{aligned}$$

$\gamma(x) = \frac{p_{X|1}(x)}{p_{X|2}(x)}$ is the likelihood ratio.

Likelihood ratio (3) – linearity, convex subset of $\mathbb{R}$

The subset of observations $X(d^*)$ for which the decision d^* should be made is the solution of the system of inequalities

$$\gamma(x)\, c_1(d^*) + c_2(d^*) \leq \gamma(x)\, c_1(d) + c_2(d)\,, \quad d \in D \setminus \{d^*\}\,.$$

- The system is linear with respect to the likelihood ratio $\gamma(x)$.
- The subset $X(d^*)$ corresponds to a convex subset of the values of the likelihood ratio $\gamma(x)$.
- As $\gamma(x)$ are real numbers, their convex subsets correspond to the numerical intervals.

Likelihood ratio (4)

Note:

There can be more than two decisions $d \in D$, $|D| > 2$ for only two states, $|Y| = 2$.

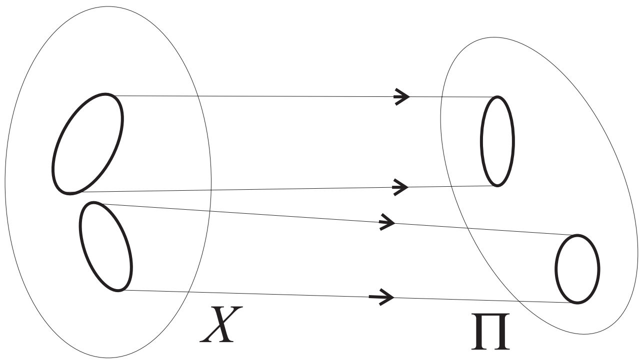
Any Bayesian strategy divides the real axis from 0 to ∞ into $|D|$ intervals $I(d)$, $d \in D$. The decision d is made for observation $x \in X$ when the likelihood ratio $\gamma = p_{X|1}(x)/p_{X|2}(x)$ belongs to the interval $I(d)$.

A more particular case which is commonly known:

Two decisions only, $D = \{1, 2\}$. Bayesian strategy is characterized by a single threshold value θ. For an observation x the decision depends only on whether the likelihood ratio is larger or smaller than θ.

Space of probabilities π, an idea

- Consider a $|Y|$-dimensional linear space Π which we call the space of probabilities.

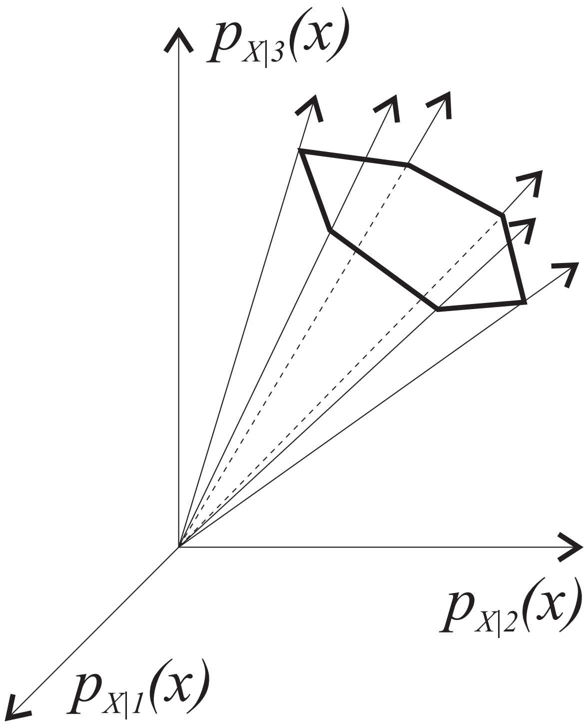


- The space of probabilities Π has coordinate axes given by probabilities $p_{X|1}(x)$, $p_{X|2}(x)$, $\ldots$ (in general $p_{X|y}(x)$, $y \in Y$).
- The set of observations X is mapped into a positive hyperquadrant of Π. The observation $y \in Y$ maps to the point $p_{X|y}(x)$, $y \in Y$.
- An interesting question: Where does the whole subset $X(d)$, $d \in D$, of the observation space corresponding to individual decisions maps in the space of probabilities Π?

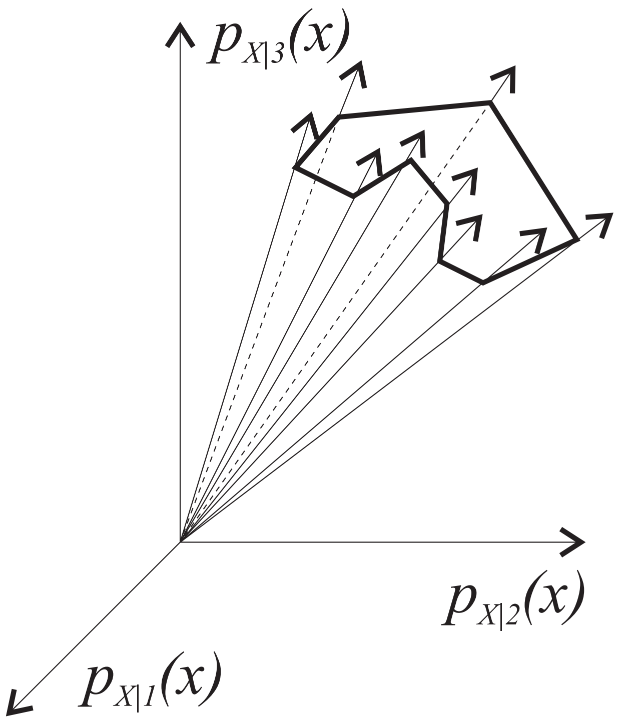
Cone, convex cone

The subset $\Pi' \subset \Pi$ is called a cone if $\alpha\,\pi \in \Pi'$ for $\forall\,\pi \in \Pi'$ and for $\forall\,\alpha \in \mathbb{R}$, $\alpha > 0$.

If the subset Π' is a cone and, in addition, it is convex then it is called a convex cone.



convex cone



non-convex cone

A general case for y convex cones, $y > 2$

Theorem:

Let X, Y, D be finite sets and let $p_{XY} \colon X \times Y \to \mathbb{R}$, $W \colon Y \times D \to \mathbb{R}$ be two functions. Let $\pi \colon X \to \Pi$ be a mapping of the set X into a $|Y|$-dimensional linear space Π (space of probabilities); $\pi(x) \in \Pi$ is a point with coordinates $p_{X|y}(x)$, $y \in Y$.

Any decomposition of the positive hyperquadrant of the space Π into $|D|$ convex cones $\Pi(d)$, $d \in D$, defines a strategy q, for which $q(x) = d$ if and only if $\pi(x) \in \Pi(d)$. Then a decomposition $\Pi^*(d)$, $d \in D$, exists such that corresponding strategy q^* minimizes a Bayesian risk

$$\sum_{x \in X} \sum_{y \in Y} p_{XY}(x, y)\, W\left(y, q(x)\right) \, .$$

Proof: Convex shape of classes in Π (1)

Let us create such cones and enumerate decision $d \in D$ by numbers $n(d)$

$$\sum_{y \in Y} p_{X|Y}(x)\, p_Y(y)\, W(y, d^*) \le \sum_{y \in Y} p_{X|Y}(x)\, p_Y(y)\, W(y, d)\,, \;\; n(d) < n(d^*)\,,$$

$$\sum_{y \in Y} p_{X|Y}(x)\, p_Y(y)\, W(y, d^*) < \sum_{y \in Y} p_{X|Y}(x)\, p_Y(y)\, W(y, d)\,, \;\; n(d) > n(d^*)\,.$$

Proof: Convex shape of classes in Π (2)

Let us use coordinates in Π, $\pi_y = p_{Y|y}(x)$. The point π with coordinates π_y, $y \in Y$, has to be mapped into the set $\Pi(d^*)$, if

$$\sum_{y \in Y} \pi_y \, p_Y(y) \, W(y, d^*) \leq \sum_{y \in Y} \pi_y \, p_Y(y) \, W(y, d) \, , \qquad n(d) < n(d^*) \, ,$$

$$\sum_{y \in Y} \pi_y \, p_Y(y) \, W(y, d^*) < \sum_{y \in Y} \pi_y \, p_Y(y) \, W(y, d) \, , \qquad n(d) > n(d^*) \, .$$

The set expressed in such a way is a cone, because if the point with coordinates π_y, $y \in Y$, satisfies the inequalities then any point with coordinates $\alpha \, \pi_y$, $\alpha > 0$, satisfies the system too.

The system of inequalities is linear with respect to variables π_y, $y \in Y$, and thus the set of its solutions $\Pi(d)$ is convex.

Importance of linear classifiers. Why?

- Theoretical importance, decomposition of the space of probabilities into convex cones.
- For some statistical models, the Bayesian or non-Bayesian strategies are implemented by linear discriminant functions.
- Some non-linear discriminant functions can be implemented as linear after straightening the feature space.
- Capacity (VC dimension) of linear strategies in an n-dimensional space is $n + 1$. Thus, the learning task is correct, i.e., strategy tuned on a finite training multiset does not differ much from the correct strategy found for a statistical model.
- Efficient algorithms exist to solve linear classification tasks.